On the Additive Structure of Quantalic λ-Calculus

Bruna Salgado
University of Minho and INESC-TEC, Portugal

Renato Neves
University of Minho and INESC-TEC, Portugal

Abstract

This work aims at extending quantalic linear λ-calculus with additive structure. The focus here will be on the additive disjunction operator $\oplus$ for it closes an important gap in previous work: the lack of methods for reasoning about 'case' statements quantitatively, fundamental across a myriad of computational paradigms.

Among other things, we extend the associated quantalic equational system to encompass the additive operator $\oplus$. We show that this extension is sound. We also show that when certain continuity properties (of the underlying quantale) are adopted it is additionally (approximately) complete. We briefly illustrate its use in probabilistic programming.

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1 Introduction

Previous work [7, 8] introduced a quantalic generalisation of linear λ-calculus, the exponential-free *multiplicative* fragment of linear logic. Here we start investigating the incorporation of *additive* structure to this body of work. Specifically our focus is on the *additive disjunction operator* $\oplus$, which is typically interpreted via coproducts and gives rise to 'case' statements (*i.e.* conditionals). Our motivation for it is highly practical: in trying to reason quantitatively about (higher-order) programs we often fell short when these involved conditionals. Of course applications involving $\oplus$ are broader than this, and typically fit in the more general pattern of reasoning quantitatively about co-Cartesian categories enriched over so-called 'generalised metric spaces' [19].

Remarkably a number of important results already considered additive structure in the quantalic setting, even if sometimes implicitly. References [14, 15, 16, 12] for example are framed in the setting of universal algebra and therefore involve additive conjunction (*i.e.* &), typically interpreted via categorical products. In the higher-order setting, [13] enforces additive conjunction to be left adjoint to implication (interpreted via Cartesian-closedness), with a series of negative results emerging from this. Our work is orthogonal to these in that we study the dual of & (*i.e.* $\oplus$) and furthermore we assume the left adjoint of implication to be multiplicative conjunction (*i.e.* $\otimes$) instead of the additive counterpart. Among other things, this removes the obstacles discussed in [13].

In this note we extend the quantalic equational system of [7, 8] to encompass the additive disjunction operator. We show that the extension is sound. We also show that when certain continuity properties (of the underlying quantale) are adopted it is complete. We show furthermore that even if the well-known Archimedean rule (often problematic) is dropped one still retains 'approximate completeness'. We briefly illustrate our extended framework in the setting of probabilistic programming [6, 3]. We will focus specifically on reasoning about Cauchy sequences of (higher-order) programs – highlighting the emergent shift from "seeing program semantics as the science of program equivalence" [13] to more flexible, quantitative perspectives, involving functional analysis and beyond.

2 Quantalic λ-calculus with additive disjunction

The extension of linear λ-calculus in [7, 8] with additive disjunction is quite simple. The grammar of types now includes the type construct $\mathbb{A} \oplus \mathbb{A}$ and the judgement formation rules are extended with those in Figure 1.

$$\frac{\Gamma \triangleright v : \mathbb{A}}{\Gamma \triangleright \mathrm{inl}_{\mathbb{B}}(v) : \mathbb{A} \oplus \mathbb{B}} \ (\mathrm{inl}) \qquad \frac{\Gamma \triangleright v : \mathbb{B}}{\Gamma \triangleright \mathrm{inr}_{\mathbb{A}}(v) : \mathbb{A} \oplus \mathbb{B}} \ (\mathrm{inr})$$

$$\frac{\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \qquad \Delta, x : \mathbb{A} \triangleright w : \mathbb{D} \qquad \Delta, y : \mathbb{B} \triangleright u : \mathbb{D} \qquad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \triangleright \mathrm{case}\ v\ \{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{D}} \ (\mathrm{case})$$

Figure 1 Judgement formation rules for the additive operator $\oplus$.

It is laborious but straightforward to prove that the extended calculus inherits desirable features from the original calculus. Most notably it inherits the *unique derivation*, *substitution*, and *exchange* properties (where as usual the latter allows to change the order of variables in contexts). These rely on a shuffling mechanism whose details can be consulted in [7, 8, 20]. The mechanism can actually be briefly glanced at in rule (case), where we stipulate that context E is a *shuffle* of the contexts Γ and Δ: in other words it is a permutation of the variables in Γ, Δ that preserves their relative order in Γ and in Δ.

In order to extend the quantalic equational system in [7, 8] with additive disjunction we need preliminaries. Thus let $\mathcal{V}$ denote a commutative and unital quantale, $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ the corresponding binary operation, and k its unit [19]. The following definition is essential for achieving our '(approximate) completeness' result.

▶ **Definition 1.** *Consider a complete lattice L. For every $x, y \in L$ we say that y is* way-below *x (in symbols, $y \ll x$) if for every subset $X \subseteq L$ whenever $x \leq \bigvee X$ there exists a* finite *subset $F \subseteq X$ such that $y \leq \bigvee F$. The lattice L is called* continuous *iff for every $x \in L$,*

$$x = \sup\{y \mid y \in L \text{ and } y \ll x\}$$

Let L be a complete lattice. A basis *B of L is a subset $B \subseteq L$ such that for every $x \in L$ the set $B \cap \{y \mid y \in L \text{ and } y \ll x\}$ is directed and has x as the least upper bound.*

We also crucially rely on the following observations. Since every quantale $\mathcal{V}$ is a cocomplete category (specifically a complete sup-lattice) it will be complete as well [1, Section 12], in other words it has all infima. Also if $\mathcal{V}$ is continuous then for every $x \in \mathcal{V}$ the operation,

$$x \wedge (-) : \mathcal{V} \to \mathcal{V}$$

is continuous as well, *i.e.* it preserves directed suprema [10, Proposition I-1.8]. Accordingly we will assume that the underlying lattice of $\mathcal{V}$ is continuous and has a basis $B \ni k$ closed under finite joins/meets and the multiplication of the quantale $\otimes$. We also assume that $\mathcal{V}$ is *integral*, *i.e.* that the unit k is the top element of $\mathcal{V}$, a common assumption in quantale theory [21]. Several examples of quantales that satisfy these constraints are presented and discussed in [7, 8]. Here we mention briefly the case of the metric quantale, for we use it in our illustration of probabilistic programming: in a nutshell, $\mathcal{V}$ is the set $[0,\infty]$ and a basis is given by the non-negative rational numbers with infinity; the operation $\otimes$ is addition, the underlying order $\leq$ of $\mathcal{V}$ is $\geq_{[0,\infty]}$, and the relation $\ll$ is the strictly greater $>$ relation with $\infty > \infty$ (thus note that in this setting the top element k is actually 0 and ∞ is the least element).

We are ready to present our quantalic equational system extended with the additive disjunction operator. In the original system, equations are labelled by elements $q \in B \subseteq \mathcal{V}$ of the quantale and classical equations $v = w$ are represented by $v =_k w$ together with $w =_k v$. The only difference is that the extended system now incorporates the rules in Figure 2. The equations on top of the dotted line are those already known for additive disjunction in the classical setting (see for example [5]). The ones on the bottom are new and serve as a form of 'quantalic congruence'. Most notably the expression $q \otimes (r \wedge s)$ between case statements encodes a form of worst-case assumption: intuitively we take the 'worst' value w.r.t. $\{r, s\}$ to reflect the possibility of taking the branch in which the two respective terms 'differ' the most – such value then compounds with q to reflect the 'difference' between the tests v and v'.

In the metric setting an equation $v =_q w$ reads as "the two terms are *at most* at distance q of each other" and the expression $q \otimes (r \wedge s)$ instantiates to $q + (r \vee s)$.

Observe that whilst the original quantalic system makes use of the quantale's linear structure (*i.e.* $\otimes$), the extended version now also make use of the quantale's Cartesian structure (*i.e.* infima). This ties up nicely with the corresponding categorical semantics, which we detail in the following section.

$$\mathrm{case}\ \mathrm{inl}_{\mathbb{B}}(v)\ \{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\} = w[v/x]$$
$$\mathrm{case}\ \mathrm{inr}_{\mathbb{A}}(v)\ \{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\} = u[v/y]$$
$$\mathrm{case}\ v\ \{\mathrm{inl}_{\mathbb{B}}(y) \Rightarrow w[\mathrm{inl}_{\mathbb{B}}(y)/x]; \mathrm{inr}_{\mathbb{A}}(z) \Rightarrow w[\mathrm{inr}_{\mathbb{A}}(z)/x]\} = w[v/x]$$

....

$$\frac{v =_q w}{\mathrm{inl}_{\mathbb{B}}(v) =_q \mathrm{inl}_{\mathbb{B}}(w)} \qquad \frac{v =_q w}{\mathrm{inr}_{\mathbb{A}}(v) =_q \mathrm{inr}_{\mathbb{A}}(w)}$$

$$\frac{v =_q v' \qquad w =_r w' \qquad u =_s u'}{\mathrm{case}\ v\ \{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\} =_{q \otimes (r \wedge s)} \mathrm{case}\ v'\ \{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w'; \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u'\}}$$

Figure 2 Quantalic equational system for additive disjunction.

3 Categorical semantics

The terms of the calculus detailed in the previous section are interpreted standardly in any symmetric monoidal closed (*i.e.* autonomous) category with binary coproducts. See a complete account for example in [5]. The interpretation of $\mathcal{V}$-equations on the other hand requires a series of preliminaries which we briefly detail next.

▶ **Definition 2.** *A $\mathcal{V}$-category is a pair (X, a) where X is a set and $a : X \times X \to \mathcal{V}$ is a function (*i.e. *a $\mathcal{V}$-relation) that satisfies:*

$$k \leq a(x_1, x_1) \qquad \textit{and} \qquad a(x_1, x_2) \otimes a(x_2, x_3) \leq a(x_1, x_3) \qquad (x_1, x_2, x_3 \in X)$$

Take two $\mathcal{V}$-categories (X, a) and (Y, b). A $\mathcal{V}$-functor $f : (X, a) \to (Y, b)$ is a function $f : X \to Y$ that satisfies the inequality $a(x_1, x_2) \leq b(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

$\mathcal{V}$-categories and $\mathcal{V}$-functors form a category which we denote by $\mathcal{V}$-Cat. A $\mathcal{V}$-category (X, a) is called *symmetric* if $a(x_1, x_2) = a(x_2, x_1)$ for all $x_1, x_2 \in X$. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ the full subcategory of $\mathcal{V}$-Cat whose objects are symmetric. Every $\mathcal{V}$-category carries a natural order defined by $x_1 \leq x_2$ whenever $k \leq a(x_1, x_2)$. A $\mathcal{V}$-category is called *separated* if its natural order is anti-symmetric. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ the full subcategory of $\mathcal{V}$-Cat whose objects are separated. When $\mathcal{V}$ is the metric quantale, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is the category Met of metric spaces and non-expansive maps. The categories $\mathcal{V}$-Cat, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$, and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ are autonomous whenever the quantale $\mathcal{V}$ is integral (see details in [7, 8]). Such gives rise to the following particular notion of enriched category.

▶ **Definition 3.** *A $\mathcal{V}$-Cat-enriched autonomous category C is an autonomous and $\mathcal{V}$-Cat-enriched category C such that the bifunctor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a $\mathcal{V}$-Cat-functor and the adjunction $(- \otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}$-Cat-adjunction. We obtain analogous notions of enriched autonomous category by replacing $\mathcal{V}$-Cat (as basis of enrichment) with $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$.*

The category $\mathcal{V}$-Cat and the aforementioned variants also have products, given precisely by the quantale's Cartesian structure (*i.e.* infima). This means that $\mathcal{V}$-Cat provides an additional basis of enrichment via products – and this is what we will recur to in the interpretation of the extended quantalic system. Specifically we will assume that the categories involved in the interpretation have binary coproducts enriched over the Cartesian structure of $\mathcal{V}$-Cat (rather than the monoidal structure). An abundance of examples of such categories is given by the following proposition, which we prove in the appendix (Section B).

▶ **Proposition 4.** *The categories $\mathcal{V}$-Cat, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$, and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ have binary coproducts enriched over their Cartesian structure.*

Next we present soundness and completeness for the interpretation structures just described. We start with the notion of (symmetric) $\mathcal{V}\lambda$-theory.

▶ **Definition 5** ($\mathcal{V}\lambda$-theories)**.** *Consider a tuple (G, Σ) consisting of a set G of ground types and a set Σ of sorted operation symbols. A $\mathcal{V}\lambda$-theory $((G, \Sigma), Ax)$ is a triple such that Ax is a set of $\mathcal{V}$-equations-in-context over λ-terms built from (G, Σ). The theory is called symmetric if it also contains the symmetry rule (see details in [7, 8]). Elements of Ax will be called axioms and equations derivable from the equational system and Ax will be called theorems.*

▶ **Definition 6** (Models of $\mathcal{V}\lambda$-theories)**.** *Consider a $\mathcal{V}\lambda$-theory $((G, \Sigma), Ax)$ and a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-enriched autonomous category C with binary coproducts enriched over the Cartesian structure of $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$. Suppose also that for each $X \in G$ we have an interpretation $[\![X]\!]$ as a C-object and analogously for the operation symbols. This interpretation structure is a model of the theory if all axioms in Ax are satisfied by the interpretation. In case the theory is symmetric we change the basis of enrichment from $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ to $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ (see details in [7, 8]).*

Take an interpretation structure as per the previous definition. We say that a $\mathcal{V}$-equation $\Gamma \triangleright v =_q w : \mathbb{A}$ holds in the interpretation if $q \leq a([\![v]\!], [\![w]\!])$ where $a : \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!]) \times \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!]) \to \mathcal{V}$ is the underlying function of the $\mathcal{V}$-category $\mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!])$.

▶ **Theorem 7** (Soundness and Completeness)**.** *Consider a $\mathcal{V}\lambda$-theory $\mathscr{T}$. A $\mathcal{V}$-equation-in-context is a theorem iff it holds in all models of the theory.*

Proof sketch. The proof piggybacks on the one in [7, 8], *i.e.* we only need to focus on the cases that involve additive disjunction. Nonetheless we still give a broad overview of the proof so that the reader gets a general feeling of what it requires.

The soundness part uses induction over the depth of proof trees that arise from the extended deductive system. The general strategy for each inference rule is to use the autonomous enrichment as well as the definition of a $\mathcal{V}$-category. The case of additive disjunction additionally requires the use of Cartesian enrichment.

Completeness on the other hand is based on the idea of a Lindenbaum-Tarski algebra. Concretely we build the syntactic category $\mathsf{Syn}(\mathscr{T})$ (also known as term model) of the underlying theory $\mathscr{T}$ and then show that provability of $\Gamma \triangleright v =_q w : \mathbb{A}$ in $\mathscr{T}$ is equivalent to $a(\llbracket v \rrbracket, \llbracket w \rrbracket) \geq q$ in the category $\mathsf{Syn}(\mathscr{T})$. Thus for two types $\mathbb{A}$ and $\mathbb{B}$, let $\mathbf{Values}(\mathbb{A},\mathbb{B})$ be the set of λ-terms v such that $x : \mathbb{A} \triangleright v : \mathbb{B}$. We equip $\mathbf{Values}(\mathbb{A},\mathbb{B})$ with the $\mathcal{V}$-relation a defined by,

$$a(x : \mathbb{A} \triangleright v : \mathbb{B}, y : \mathbb{A} \triangleright w : \mathbb{B}) = \sup \{q \mid v =_q w[x/y] \text{ is a theorem of } \mathscr{T}\}$$

It is easy to see that $\mathbf{Values}((\mathbb{A},\mathbb{B}), a)$ is a $\mathcal{V}$-category. We then quotient it into a separated $\mathcal{V}$-category via the construction detailed in [7, 8]. The next step is to prove that this quotienting procedure is compatible with the term formation rules of the extended calculus. To this effect, in general one uses the fact that $\otimes$ distributes over suprema and the case of additive disjunction additionally requires the fact that $q \wedge (-)$ distributes over directed suprema for every $q \in \mathcal{V}$. This yields the desired category $\mathsf{Syn}(\mathscr{T})$ which will respect Definition 3 and moreover possess binary coproducts enriched over the Cartesian structure of $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$ (resp. $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}$).

The final step is to show that if an equation $\Gamma \triangleright v =_q v' : \mathbb{A}$ holds in $\mathsf{Syn}(\mathscr{T})$ then it is a theorem of $\mathscr{T}$. By assumption $a([v], [v']) = a(v, v') = \sup \{r \mid v =_r v'\} \geq q$. It follows from the definition of the way-below relation that for all $x \in B$ with $x \ll q$ there exists a finite set $F \subseteq \{r \mid v =_r v'\}$ such that $x \leq \sup F$. Then by an application of rule (join) ([8, Figure 4]) we obtain $v =_{\sup F} v'$, and consequently, rule (weak) ([8, Figure 4]) provides $v =_x v'$ for all $x \ll q$. Finally by an application of rule (arch) ([8, Figure 4]) we deduce that $v =_q v'$ is part of the theory. ◀

Whilst extremely useful, the well-known Archimedean rule (see [7, 8, 14]) (arch) has the drawback of involving infinitely many premisses. It is thus often desirable to drop it, for computational reasons. The following result tells that such rule can be dropped while retaining a weaker form of completeness.

▶ **Theorem 8** (Approximate completeness). *Consider a $\mathcal{V}\lambda$-theory $\mathscr{T}$. If $\Gamma \triangleright v =_q w : \mathbb{A}$ holds in all models of the theory then for all approximations $r \ll q$ with $r \in B$ we have $\Gamma \triangleright v =_r w : \mathbb{A}$ as a theorem. In particular if q is compact (*i.e. *$q \ll q$) we have $\Gamma \triangleright v =_q w : \mathbb{A}$.*

Proof. One just needs to remove the last sentence of the previous proof. ◀

4 A brief illustration with probabilistic programming

We now briefly illustrate our framework in the setting of probabilistic programming, using as basic examples two main topics in probability theory [9] – probabilistic predicates and random walks on the real line. Our illustration will be grounded on a standard probabilistic model, namely the category Ban of Banach spaces and linear contractions [6]. As discussed in [7, 8] this category has a Met-enriched autonomous structure, and it is well-known that it has binary coproducts given by the direct sum $\oplus$ equipped with the ℓ_1 norm. Thus in order to fit Ban in our framework we only need to show that its coproduct structure is enriched over the Cartesian structure of Met. We detail this in the appendix (Section A) where we also recall some basic facts about measure theory.

We proceed by presenting a metric λ-theory (Definition 5) on which to reason about predicates and random walks, as previously discussed. Our only ground type will be $\mathtt{real}$ to represent measures over real numbers – *i.e.* we set $[\![\mathtt{real}]\!]$ to be the space $\mathrm{M}(\mathbb{R})$ of measures over the real line. Recall that the monoidal unit of Ban is $\mathbb{R}$. Concerning operations we take a pre-determined set of predicates $p : \mathtt{real} \to \mathbb{I} \oplus \mathbb{I}$ whose interpretation takes the form $[\![p]\!]\,(\mu) = (\mu(U), \mu(\overline{U})) \in \mathbb{R} \oplus \mathbb{R}$ for some measurable subset of $U \subseteq \mathbb{R}$. Intuitively $U \subseteq \mathbb{R}$ corresponds to the subspace in which the predicate is supposed to hold. We also take a pre-determined set of actions $a : \mathbb{I} \to (\mathbb{A} \multimap \mathbb{A})$ and a pre-determined set of measures $m : \mathbb{I} \to \mathtt{real}$ whose interpretation takes no particular form. Finally we consider addition $+ : \mathtt{real}, \mathtt{real} \to \mathtt{real}$ whose interpretation is given by $\mu \otimes \nu \mapsto +_*(\mu \otimes \nu)$ where $+_*$ is the pushforward measure construction of $+$ (see further details in [7, 8]). Next, given a measure m and actions a, b consider the following 'abstract' Bernoulli trial,

$$p : \mathtt{real} \multimap \mathbb{I} \oplus \mathbb{I} \rhd \underbrace{\mathrm{case}\ p(m(*))\ \mathrm{of}\ \mathrm{inl}(x) \Rightarrow a(x); \mathrm{inr}(y) \Rightarrow b(y)}_{\mathtt{bern}(p)} : \mathbb{A} \multimap \mathbb{A}$$

Note that if the metric equation $p_1(m(*)) =_\epsilon p_2(m(*))$ holds for two predicates p_1, p_2 : $\mathtt{real} \to \mathbb{I} \oplus \mathbb{I}$ then the equation $\mathtt{bern}(\lambda x.p_1(x)) =_\epsilon \mathtt{bern}(\lambda x.p_2(x))$ must hold as well (as per our equational system). Such is useful to approximate Bernoulli trials that may be hard to compute as illustrated by the following examples.

▶ **Example 9** (Predicates and Cauchy sequences)**.** Take a measure m and the predicate,

$$x : \mathtt{real} \rhd p_{\frac{1}{2}\sqrt{2}}(x) : \mathbb{I} \oplus \mathbb{I}$$

that returns true if $x < \frac{1}{2}\sqrt{2}$ and false otherwise. Given the irrationality of $\frac{1}{2}\sqrt{2}$ it is natural to consider successive approximations $(-) \rhd p_{q_n}(m(*)) : \mathbb{I} \oplus \mathbb{I}$ $(n \in \mathbb{N})$ in which the condition $x < \frac{1}{2}\sqrt{2}$ is replaced by $x < q_n$ for q_n a rational number. We show next how our framework makes this idea precise. Take a sequence of rational numbers $(q_n)_{n\in\mathbb{N}}$ that converges to $\frac{1}{2}\sqrt{2}$ from below. We then postulate as axioms in our deductive system that $(p_{q_n}(m(*)))_{n\in\mathbb{N}}$ is a Cauchy sequence and furthermore that it converges to $p_{\frac{1}{2}\sqrt{2}}(m(*))$. Such is asserted precisely by setting,

$$\begin{cases} \forall \epsilon > 0.\, \exists k \in \mathbb{N}.\, \forall n \geq k.\, p_{q_n}(m(*)) =_\epsilon p_{q_{n+1}}(m(*)) & \text{(Cauchy sequence)} \\ \forall \epsilon > 0.\, \exists k \in \mathbb{N}.\, \forall n \geq k.\, p_{q_n}(m(*)) =_\epsilon p_{\frac{1}{2}\sqrt{2}}(m(*)) & \text{(Convergence)} \end{cases} \tag{1}$$

for appropriate choices of k (which in our context is irrelevant to detail). The next step is to prove that this axiomatics is sound, *i.e.* that such equations hold in Ban, which is detailed in the appendix. In the next example we capitalise on such approximations, now formulated precisely, to reason about approximations of random walks.

▶ **Example 10** (Random walk approximations)**.** We now consider the λ-term,

$$(-) \rhd \underbrace{\lambda x_1. \ldots x_k.\, y.\, x_1(\ldots(x_k(y))\ldots)}_{\mathtt{sequence_k}}$$

which operationally speaking sequences k terms given as input. Also given a predicate $p : \mathtt{real} \to \mathbb{I} \oplus \mathbb{I}$, take the term $(-) \rhd \mathtt{sequence_k}\ \mathtt{bern}(\lambda x.\, p(x)) \ldots \mathtt{bern}(\lambda x.\, p(x)) : \mathbb{A} \multimap \mathbb{A}$

which intuitively represents an *abstract random walk of k-steps*. In order to keep our notation simple we abbreviate this last term to $\texttt{rwalk}(\lambda x.\, p(x))$. Now, it follows from our system that if $p_1(m(*)) =_\epsilon p_2(m(*))$ for two predicates p_1 and p_2 and a measure m then,

$$\texttt{rwalk}(\lambda x.p_1(x)) =_{k\cdot\epsilon} \texttt{rwalk}(\lambda x.p_2(x))$$

In particular, from the previous example we deduce that $\texttt{rwalk}(\lambda x.\, p_{q_n}(x))$ is a Cauchy sequence that converges to $\texttt{rwalk}(\lambda x.\, p_{\frac{1}{2}\sqrt{2}}(x))$. In other words the approximations obtained in the previous example propagate to the corresponding random walks.

As a final illustration of the synergy between syntax and semantics that our framework provides, suppose now that the actions $a, b : \mathbb{I} \to \mathbb{A} \multimap \mathbb{A}$ involved in $\texttt{bern}(\lambda x.\, p(x))$ are concrete jumps on the real line. To this effect we set the interpretations $[\![a]\!], [\![b]\!] : [\![\mathbb{I}]\!] \to [\![\texttt{real}]\!] \multimap [\![\texttt{real}]\!]$ to be,

$$[\![a]\!](1) = \mu \mapsto +_*(\mu \otimes \texttt{unif}(0,1)) \qquad [\![b]\!](1) = \mu \mapsto +_*(\mu \otimes \texttt{unif}(-1,0))$$

where $\texttt{unif}(0,1) \in \mathrm{M}(\mathbb{R})$ is the uniform distribution on the interval $[0,1]$ and analogously for $\texttt{unif}(-1,0)$. Operationally a corresponds to a jump to the right with magnitude between 0 and 1, and analogously for b. Suppose we have another action $c : \mathbb{I} \to (\texttt{real} \multimap \texttt{real})$ whose interpretation is that of a except for the fact that $\texttt{unif}(0,1)$ is replaced by $\texttt{unif}(0,1+\delta)$. What will be the effect on the random walk when replacing a by c? Our approach for answering the previous question starts by 'decomposing' the actions a and c, via the axioms,

$$a(*) =_0 \lambda z.\ +(z, \texttt{unif}(0,1)(*)) \qquad c(*) =_0 \lambda z.\ +(z, \texttt{unif}(0,1+\delta)(*))$$

whose soundness is straightforward to prove. Note that we are slightly abusing notation by using $\texttt{unif}(0,1)$ (resp. $\texttt{unif}(0,1+\delta)$) both as syntactic and semantic objects.

The next step is to observe that one can put an upper bound between $a(*)$ and $c(*)$ via the previous axioms and an upper bound between the terms $\texttt{unif}(0,1)(*)$ and $\texttt{unif}(0,1+\delta)(*)$. The latter upper bound is obtained *semantically* by computing the norm $\|\texttt{unif}(0,1) - \texttt{unif}(0,1+\delta)\|$ in the way described in Section A. It will be specifically $2 \cdot \frac{\delta}{1+\delta}$. Then as our final step we proceed *syntactically* via the system of [7, 8] extended with additive disjunction, as follows.

$$\begin{aligned}
&\text{case } p(m) \text{ of } \mathrm{inl}(x) \Rightarrow a(x); \mathrm{inr}(y) \Rightarrow b(y)\\
&=_0 \text{case } p(m) \text{ of } \mathrm{inl}(x) \Rightarrow x \text{ to } * \,.\, a(*); \mathrm{inr}(y) \Rightarrow b(y)\\
&=_{2\cdot\frac{\delta}{1+\delta}} \text{case } p(m) \text{ of } \mathrm{inl}(x) \Rightarrow x \text{ to } * \,.\, c(*); \mathrm{inr}(y) \Rightarrow b(y)\\
&=_0 \text{case } p(m) \text{ of } \mathrm{inl}(x) \Rightarrow c(x); \mathrm{inr}(y) \Rightarrow b(y)
\end{aligned}$$

Thus if $\texttt{rwalk}(\lambda x.p(x))$ is the random walk that involves action a and $\texttt{rwalk}'(\lambda x.p(x))$ the random walk that involves action c we deduce from the framework the metric equation,

$$\texttt{rwalk}(\lambda x.p(x)) =_{2k\cdot\frac{\delta}{1+\delta}} \texttt{rwalk}'(\lambda x.p(x))$$

which will converge to 0 as δ tends to 0.

5 Current work

We are currently exploring the application of our framework to three other computational paradigms that we find to be particularly enticing. Namely, quantum computation [17], stochastic hybrid computation [11], and synthetic guarded domain theory [4]. Whilst the first two cases involve the metric quantale (as in the probabilitic example) the third case involves the so-called ultrametric quantale. For all three cases we are currently in the process of identifiying and/or building models that fit the requirements demanded from our framework.

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A Support material for the probabilistic illustration

The autonomous Met-enrichment of Ban is induced by the *operator norm*. Specifically given a linear map $T : V \to W$ between Banach spaces we have,

$$\|T\| = \sup\{\|T(v)\| \mid v \in V, \|v\| = 1\}$$

Linear contractions will be precisely those linear maps T such that $\|T\| \leq 1$, and the distance between two contractions T and S is set as $\|T - S\|$. Given $T : V \to W$ and $S : U \to W$ their co-pairing $[T, S] : V \oplus U \to W$ is defined by $[T, S](v, u) = T(v) + S(u)$. The fact that the operator $[T, S]$ is contractive follows from the inequation $\|[T, S]\| \leq \max\{\|T\|, \|S\|\}$ – which is straightforward to prove when one notices that every unitary vector $(v, u) \in V \oplus U$ can be rewritten as,

$$\left(\|v\| \frac{1}{\|v\|} v, \|u\| \frac{1}{\|u\|} u\right) \qquad \|v\| + \|u\| = 1$$

The fact that the coproduct structure of Ban is enriched over the Cartesian structure of Met then follows rather directly,

$$\begin{aligned} d([T, S], [T', S']) &= \|[T, S] - [T', S']\| \\ &= \|[T - T', S - S']\| \\ &\leq \max\{\|T - T'\|, \|S - S'\|\} \\ &= \max\{d(T, T'), d(S, S')\} \end{aligned}$$

Our illustration involves the notion of a measure which we briefly describe next (see *e.g.* [2, Chapter 10] or [18, Chapter 2] for a thorough account).

▶ **Definition 11.** *For a measurable space (X, Σ_X) a measure is a function $\mu : \Sigma_X \to \mathbb{R}$ such that $\mu(\emptyset) = 0$ and moreover it is σ-additive,* i.e.

$$\mu\left(\bigcup_{i=1}^{\infty} U_i\right) = \sum_{i=1}^{\infty} \mu(U_i)$$

where $(U_i)_{i \in \omega}$ is any family of pairwise disjoint measurable sets. A measure μ is called positive if $\mu(U) \geq 0$ for all measurable sets U and a distribution if furthermore $\mu(X) = 1$.

For a measurable space X the set of measures $\mathrm{M}(X)$ forms a vector space via pointwise extension. It also forms a Banach space when equipped with the total variation norm,

$$\|\mu\| = \sup\left\{\sum_{i=1}^{n} \|\mu(U_i)\| \mid \{U_1, \dots, U_n\} \text{ is a measurable partition}\right\}$$

In our probabilistic illustration we axiomatised that $\texttt{bern}(\lambda x.\, p_{q_n}(x))_{n \in \mathbb{N}}$ is a Cauchy sequence that furthermore converges to $\texttt{bern}(\lambda x.\, p_{\frac{1}{2}\sqrt{2}}(x))$. We show next that this axiomatics is sound, via the following reasoning.

$$
\begin{aligned}
& \left[\!\left[p_{\frac{1}{2}\sqrt{2}}(x) \right]\!\right](\mu) \\
&= \left(\mu \left(-\infty, \frac{1}{2}\sqrt{2} \right), \mu(X) - \mu \left(-\infty, \frac{1}{2}\sqrt{2} \right) \right) \\
&= \left(\mu \left(\bigcup_{n \in \mathbb{N}} (-\infty, q_n) \right), \mu(X) - \mu \left(-\infty, \frac{1}{2}\sqrt{2} \right) \right) && \left\{ (q_n)_{n\in\mathbb{N}} \nearrow \frac{1}{2}\sqrt{2} \right\} \\
&= \left(\sup_{n \in \mathbb{N}} \mu \left((-\infty, q_n) \right), \mu(X) - \mu \left(-\infty, \frac{1}{2}\sqrt{2} \right) \right) && \{\text{Measure properties}\} \\
&= \left(\lim_{n\to\infty} \mu \left((-\infty, q_n) \right), \mu(X) - \mu \left(-\infty, \frac{1}{2}\sqrt{2} \right) \right) && \{\text{Limits coincide with sup. of inc. seq.}\} \\
&= \left(\lim_{n\to\infty} \mu \left((-\infty, q_n) \right), \lim_{n\to\infty} \mu \overline{(-\infty, q_n)} \right) && \{\text{Measure properties}\} \\
&= \lim_{n\to\infty} \left(\mu \left((-\infty, q_n) \right), \mu \overline{(-\infty, q_n)} \right) \\
&= \lim_{n\to\infty} [\![p_{q_n}(x)]\!](\mu)
\end{aligned}
$$

Finally a useful fact about computing norms is that $\|\mu\| = \mu^+(X) + \mu^-(X)$ where μ^+ and μ^- are the positive and negative parts of μ respectively (see details in [2, Section 8.2. and Section 10.10]). We use this to compute the norm of $\texttt{unif}(0,1) - \texttt{unif}(0, 1+\delta)$, as required in the main text. First,

$$
\begin{aligned}
& \|\texttt{unif}(0,1) - \texttt{unif}(0,1+\delta)\| \\
&= (\texttt{unif}(0,1) - \texttt{unif}(0,1+\delta))^+(\mathbb{R}) + (\texttt{unif}(0,1) - \texttt{unif}(0,1+\delta))^-(\mathbb{R})
\end{aligned}
$$

and proceed by computing the left-hand side of the addition,

$$
\begin{aligned}
& (\texttt{unif}(0,1) - \texttt{unif}(0,1+\delta))^+(\mathbb{R}) \\
&= \sup\{\texttt{unif}(0,1)(U) - \texttt{unif}(0,1+\delta)(U) \mid U \subseteq \mathbb{R}\} \\
&= \sup\{\texttt{unif}(0,1)(U \cap [0,1]) - \texttt{unif}(0,1+\delta)(U \cap [0,1]) - \texttt{unif}(0,1+\delta)(U \cap (1, 1+\delta]) \mid U \subseteq \mathbb{R}\} \\
&= \sup\left\{ \left(1 - \frac{1}{1+\delta}\right) \texttt{unif}(0,1)(U \cap [0,1]) - \texttt{unif}(0,1+\delta)(U \cap (1,1+\delta]) \mid U \subseteq \mathbb{R} \right\} \\
&= 1 - \frac{1}{1+\delta}
\end{aligned}
$$

It follows from an analogous reasoning the right-hand side of the addition will be $\frac{\delta}{1+\delta}$ and therefore the norm will be $2 \cdot \frac{\delta}{1+\delta}$.

B Proofs

Proof of Proposition 4. Given two $\mathcal{V}$-categories (X, a_X) and (Y, a_Y) the carrier of the coproduct is given by the Set-theoretic coproduct $X + Y$. The corresponding $\mathcal{V}$-relation a_{X+Y} is given by,

$$
\begin{cases}
a_{X+Y}(\mathrm{inl}(x_1), \mathrm{inl}(x_2)) = a_X(x_1, x_2) \\
a_{X+Y}(\mathrm{inr}(y_1), \mathrm{inr}(y_2)) = a_Y(y_1, y_2) \\
a_{X+Y}(\mathrm{inl}(x), \mathrm{inr}(y)) = a_{X+Y}(\mathrm{inr}(y), \mathrm{inl}(x)) = \bot
\end{cases}
$$

Co-pairing is defined as in Set. The Cartesian enrichment follows straightforwardly from the fact that the $\mathcal{V}$-relation of every hom-set $\mathcal{V}\text{-}\mathsf{Cat}((X, a_X), (Y, a_Y))$ is given by infima and the equation $(\bigwedge \mathscr{D}) \wedge (\bigwedge \mathscr{F}) = \bigwedge(\mathscr{D} \cup \mathscr{F})$ for all subsets $\mathscr{D}, \mathscr{F}$ of $\mathcal{V}$. The same reasoning about the Cartesian enrichment applies to $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$ and $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}$. ◀