Towards an Axiomatisation of the Neighborhood Monad

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Abstract

We propose a framework of universal algebra for specifying graded monads on the category of pseudometric spaces and nonexpansive maps with grades in the additive preordered monoid of extended positive reals. The ensuing concept of graded quantitative theory combines ideas from the work on universal algebra over relational Horn models and the graded algebraic theories of Kura. Our key motivation is to develop a presentation of the so-called neighborhood monad that features in the denotational semantics of Numerical Fuzz, a linear type system for tracking bounds on the rounding error incurred in numerical programs. In this direction, we introduce the graded quantitative theory of neighbors and prove that its associated variety admits free algebras, which enables us to associate a graded monad with the theory. We leave it open for future work to establish the precise relation between the neighborhood monad and our graded quantitative theory of neighbors.

2012 ACM Subject Classification Theory of computation → Program semantics; Mathematics of computing → Numerical analysis

Keywords and phrases quantitative algebra, graded monad, forward error analysis

Funding *Max Fan*: NSF Graduate Research Fellowship under Award #2139899
Jonas Forster: Deutsche Forschungsgemeinschaft (DFG) – project number 434050016
Justin Hsu: NSF Award #2153916 and a Royal Society Wolfson Visiting Fellowship
Jessica Richards: DeepMind Graduate Scholarship

Acknowledgements We would like to thank the organisers of the ACT Adjoint School 2025 for bringing together the authors and enabling this collaboration. This work also benefited from discussions with Aven Dauz and Laura Zielinski.

1 Introduction

When writing numerical programs, we would ideally wish to compute over the reals. Since this is technically not feasible, most programs utilize floating point numbers, a discrete finitary approximation of the reals. A central task for numerical analysis is to develop methods that bound the difference between ideal computations (i.e. true arithmetic) and the approximate computations carried out by computer programs (i.e. floating point arithmetic). This difference is known as *round-off error*.

Numerical Fuzz [14] is a bounded linear type system capable of tracking round-off error. This is achieved by incorporating a graded monadic type $T_q\mathbf{num}$ where the grade q is a point in the preordered monoid formed by the extended non-negative reals under addition encoding the greatest possible round-off error. That is, for a program e of type $T_q\mathbf{num}$, the difference evaluating e over the reals and the floats is bounded above by q.

Monadic types are typically interpreted as monads in their denotational semantics. Graded monads [11] refine this concept by a stratification of the monad structure dictated by a small monoidal category. In the denotational semantics of Numerical Fuzz, the monadic type T is modeled by the *graded neighborhood monad*, where grades are extended non-negative reals.

In the algebraic effects literature, ungraded computational effects can be presented by corresponding algebraic theories. Effects are then combined through tensoring. This construction provides a well established framework for combining algebraic effects.

Much like finitary monads on **Set** correspond to varieties of finitary algebras [12], graded monads on **Set** admit presentations by means of graded equational theories [16]. However, for base categories beyond **Set**, there does not seem to be a corresponding framework of graded universal algebra except in the case of grades in the additive monoid of natural numbers [8]. We aim to partially fill this gap by describing the first concept of an $\mathbf{R}$-graded quantitative theory, where $\mathbf{R}$ denotes the preordered additive monoid of the extended non-negative reals. In future work, we hope to systematically combine the neighborhood monad with effects using this new syntactic technology. Existing work has done this on an ad-hoc basis for specific effects, which we summarize here for purpose of illustration:

1. The *maybe* monad for modeling so-called *denormal computations.* Denormal computation occurs when a computation falls below the least representable float in hardware.
2. The *powerset* monad for non-deterministic rounding. This can occur when the IEEE floating-point standard leaves the result of a floating-point operation underspecified. Concretely, this occurs in the case of ties. For example, when the round-to-nearest float mode is set and the ideal computation sits exactly at the midpoint between nearby floats.
3. The *finite distribution* monad for randomized rounding, which can model rounding occuring outside of the IEEE floating-point standard. For example, randomized rounding is useful in the design of various approximation algorithms.

Together, these extensions enable Numerical Fuzz to more accurately and conveniently model real-world computation. We hasten to reemphasize that, importantly, a generalized framework for composing, e.g. the neighborhood monad with computational effects, seems to be missing. We hope that our graded quantitative theories might aid in this direction.

Inspired by the success of algebraic effects, we are interested in providing a similar algebraic recipe for combining the graded neighborhood monad with arbitrary effects. Towards this goal, we develop the first ingredient in our recipe: a graded quantitative algebraic theory of neighbors. It remains an open problem to determine whether our theory yields a presentation of the neighborhood monad in the sense that the corresponding variety of graded algebras attached to our theory coincides with the algebras of the neighborhood monad.

Related work. The neighborhood monad employed in this work was introduced by Kellison and Hsu [14], along with several variants that incorporate specific choices of a monad modelling effects [23,24]. Quantitative algebra [17] is an active area of research [1,3–6,18,20,21] aimed at extending the techniques of universal algebra to the study of algebraic structures over metric spaces. Quantitative algebra has been extended to allow for algebraic reasoning over categories of relational Horn models [9], as well as algebras with operations that fail to be nonexpansive [22]. What seems to be missing in this setting is an account of *graded* algebraic reasoning, i.e. a counterpart of graded monads [11,25] on metric spaces: we are only aware of the notion of graded quantitative theories of Ford [8] where grading is confined to the additive monoid of natural numbers (also see [10]). With that being said, equational presentations of graded monads on **Set** have been well developed at various levels of generality, e.g. [13,16,19].

2 Background

We briefly review (extended) pseudometric spaces and graded monads [11,25].

2.1 Extended Pseudometric Spaces

We denote by $\mathbb{R}_\infty$ the set of extended non-negative reals, i.e. $\mathbb{R}_\infty := \mathbb{R}^{\geq 0} \sqcup \{\infty\}$, and we extend the usual addition and ordering on the reals to $\mathbb{R}_\infty$ by defining $r + \infty = \infty$ and $r \leq \infty$ for all $r \in \mathbb{R}_\infty$. An *extended pseudometric space* is a set X equipped with a map $d : X \times X \to \mathbb{R}_\infty$ such that

$$d(x, y) = d(y, x), \qquad d(x, z) \leq d(x, y) + d(y, z), \quad \text{and} \qquad d(x, x) = 0$$

for all $x, y, z \in X$. That is, an extended pseudometric space is a carrier set together with an assignment of a *distance* $d(x, y) \in \mathbb{R}_\infty$ to each pair of points described axiomatically as above. We also write $X \models x =_r y$ to indicate that $d(x, y) \leq r$. A *non-expansive* map from (X, d_X) to (Y, d_Y) is a map $f : X \to Y$ such that $d_Y(f(x), f(x')) \leq d_X(x, x')$. We write **pMet** for the category of extended pseudometric spaces and non-expansive maps. For brevity, we omit the word extended and refer to these simply as pseudometric spaces. We sometimes notationally conflate a pseudometric space with its carrier, i.e. write X instead of (X, d_X). Given $r \in \mathbf{R}$, we write $\mathbf{2}_r$ for the pseudometric on $\{0, 1\}$ with $d(0, 1) = r$.

The category **pMet** is locally ω_1-presentable [2] as a closed category in the sense of Kelly [15]. This means that the full subcategory $\mathbf{Pres}_{\omega_1}(\mathbf{pMet})$ of **pMet** spanned by the internally countably presentable spaces, i.e. spaces X such that $[X, -]$ preserves ω_1-filtered colimits, is essentially small and every space is an ω_1-filtered colimit of such spaces. Here, the internal hom $[X, Y]$ from X to Y equips $\mathbf{pMet}(X, Y)$ with the supremum metric:

$$d_{[X,Y]}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

See, e.g., Ford et al. [9, Section 3] for details.

2.2 Graded Monads

Fix a small strict monoidal category $\mathbf{M}$ with tensor bifunctor $\otimes : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ and unit object I. An $\mathbf{M}$-*graded monad* on a category $\mathscr{A}$ is a lax monoidal functor $\mathbf{M} \to [\mathscr{A}, \mathscr{A}]$, where $[\mathscr{A}, \mathscr{A}]$ is the monoidal category of endofunctors on $\mathscr{A}$, where the tensor is composition and the unit object is the identity functor.

Hereafter, we work exclusively with $\mathbf{R}$-graded monads where $\mathbf{R}$ is the preordered monoid $(\mathbb{R}_\infty, +, 0, \leq)$, viewed as a (small and strict) monoidal category. Slightly overloading notation, we write $q \leq r$ for the unique morphism $q \to r$ whenever $q \leq r$ holds in $\mathbf{R}$. Unfolding definitions, an $\mathbf{R}$-graded monad on $\mathscr{A}$ is a functor $\mathbb{M} : \mathbf{R} \to [\mathscr{A}, \mathscr{A}]$, yielding a family of endofunctors $(M_r : \mathscr{A} \to \mathscr{A})_{r \in \mathbf{R}}$, equipped with natural transformations

$$\eta : \mathsf{Id}_{\mathscr{A}} \to M_0 \text{ (the } \textit{unit}) \qquad \text{and} \qquad \mu^{q,r} : M_q M_r \to M_{q+r} \text{ (the } \textit{multiplication}) \ (q, r \in \mathbb{R}_\infty)$$

These data are subject to axioms which resemble those of ordinary monads, up to insertion of grades. In particular, for all $q, r, s \in \mathbb{R}_\infty$ we have the unit laws and associativity laws

$$\mu^{q,0} \cdot M_q \eta_X = \mathsf{id}_{M_0 X} = \mu^{0,q} \cdot \eta_{M_q X} \qquad \text{and} \qquad \mu^{q,r+s} \cdot M_q \mu^{r,s}_X = \mu^{q+r,s}_X \cdot \mu^{q,r}_{M_s X}$$

Algebras of a graded monad. Graded monads enjoy a compatible notion of graded algebra [11], generalizing the Eilenberg-Moore algebras of ordinary monads. Fix an $\mathbf{R}$-graded monad $\mathbb{M}$ on a category $\mathscr{A}$. An *Eilenberg-Moore algebra* of $\mathbb{M}$ (or an $\mathbb{M}$-*algebra*) is a functor $A\colon \mathbf{R} \to \mathscr{A}$ (the *carrier*) equipped with a family of $\mathscr{A}$-morphisms

$$a_{q,r}\colon M_q A_r \to A_{q+r} \quad \text{(the } \textit{structure}\text{)} \qquad (q, r \in \mathbb{R}_\infty)$$

This data is subject to axioms which are, again, similar to those of regular monad algebras with added indices: For all $q, r, s \in \mathbb{R}_\infty$ we require

$$a^{0,r} \cdot \eta_{A_r} = \mathsf{id}_{A_r} \qquad \text{and} \qquad a_{q+r,s} \cdot \mu^{q,r}_{A_s} = a_{q,r+s} \cdot M_q a_{r,s}$$

Thus, the *carrier* of an $\mathbb{M}$-algebra consists of an $\mathbb{R}_\infty$-indexed family of objects $A_r \in \mathscr{A}$ with a specified morphism $A(q \leq r)\colon A_q \to A_r$ for all $q, r \in \mathbb{R}_\infty$ with $q \leq r$. A *homomorphism* from A to B is a natural transformation $h\colon A \to B$ such that

$$b_{r,s} \cdot M_r h_s = h_{r+s} \cdot a_{r,s}$$

for all $r, s \in \mathbb{R}_\infty$. We write $\mathbf{EM}(\mathbb{M})$ for the category of $\mathbb{M}$-algebras and their homomorphisms. The assignment of an $\mathbb{M}$-algebra A to its 0-part A_0 is the object-part of a forgetful functor $V\colon \mathbf{EM}(\mathbb{M}) \to \mathscr{A}$ which sends a homomorphism $h\colon A \to B$ to h_0.

3 The Neighborhood Monad

We briefly gather the necessary background on the *neighborhood monad* [14, Section 4.2]. Given $X \in \mathbf{pMet}$ and $r \in \mathbb{R}_\infty$, write

$$T_r X = \{(a, b) \in X \times X \mid d_X(a, b) \leq r\}$$

for the set of r-*neighbors*. Note that $T_\infty X = X \times X$ and $T_q X \subseteq T_r X$ if $q \leq r$. Then $T_r X$ admits the structure of a pseudometric space with the distance map defined by $d((a, b), (a', b')) := d_X(a, a')$. Indeed, one readily verifies symmetry, reflexivity, and the triangle inequality by exploiting the corresponding laws of d_X.

The assignment $X \mapsto T_r X$ is the object-part of a functor $T_r\colon \mathbf{pMet} \to \mathbf{pMet}$ with the action on a non-expansive map $f\colon X \to Y$ defined by

$$T_r f\colon T_r X \to T_r Y \qquad (a, b) \mapsto (f(a), f(b)).$$

Then $T_r f$ is defined since $(a, b) \in T_r X$ implies $d_Y(f(a), f(b)) \leq d(a, b) \leq r$ because f is non-expansive. Moreover, $T_r f$ is non-expansive, as

$$d_{T_r Y}(T_r f(a, b), T_r f(c, d)) = d_Y(f(a), f(c)) \leq d_X(a, c) = d_{T_r X}((a, b), (c, d)).$$

Kellison and Hsu [14] explain that the family $(T_r)_{r \in \mathbb{R}_\infty}$ carries the structure of an $\mathbf{R}$-graded monad, the so-called *neighborhood monad*, which we recall in the following.

▶ **Definition 3.1.** The *neighborhood monad* (notation: Nb) is the $\mathbf{R}$-graded monad obtained by equipping the family of functors $(T_r\colon \mathbf{pMet} \to \mathbf{pMet})_{r \in \mathbb{R}_\infty}$ with the unit $\eta\colon X \to T_0 X$ given by the assignment $x \mapsto (x, x)$ and multiplication defined for all $q, r \in \mathbb{R}_\infty$ by

$$\mu^{q,r}\colon T_q T_r X \to T_{q+r} X \qquad ((a, b), (c, d)) \mapsto (a, d).$$

Note that $T_r X$ and $[\mathbf{2}_r, X]$ have isomorphic underlying sets but carry distinct pseudometrics.

Algebras of the neighborhood monad. An algebra of the neighborhood monad is of a functor $A\colon \mathbf{R} \to \mathbf{pMet}$ and a family of non-expansive maps $a_{r,s}\colon T_r A_s \to A_{r+s}$, natural in r, s, satisfying $a_{0,r}(e, e) = e$ for all $e \in A_r$ and

$$a_{r,s+t}(h_{s,t}(a, b), h_{s,t}(c, d)) = a_{r+s,t}(a, d)$$

for $a, b, c, d \in A_t$ with $d(a, b) \leq s, d(c, d) \leq s$ and $d(a, c) \leq r$.

These identities closely resemble the axioms of the rectangular bands of Clifford [7]; we leave it to future work to better understand the relationship with rectangular bands.

4 Graded Quantitative Theories

We proceed to describe our notion of an $\mathbf{R}$-graded quantitative theory, which combines ideas from the finitary relational algebraic theories of Ford et al. [9] with the graded algebraic theories of Kura [16]. Crucially, we employ finite pseudometric spaces as the arities of operations. In this direction, we pick a small skeleton $\mathscr{P}_\omega \cong \mathbf{Pres}_\omega(\mathbf{pMet})$ of *arities*. For convenience, we assume that each arity is carried by a finite cardinal.

▶ **Definition 4.1.** An $\mathbf{R}$-*graded signature* consists of a set Σ of *operation symbols* together with the assignment of a *grade* $\mathsf{g}(\sigma) \in \mathbf{R}$ and an *arity* $\mathsf{ar}(\sigma) \in \mathscr{P}_\omega$ to each $\sigma \in \Sigma$. An operation σ is *n-ary* if $|\,\mathsf{ar}(\sigma)| = [n]$. A Σ-*algebra* is then a functor $A\colon \mathbf{R} \to \mathbf{pMet}$ equipped with a family of nonexpansive maps $\sigma^A = (\sigma^A_r\colon [\mathsf{ar}(\sigma), A_r] \to A_{\mathsf{g}(\sigma)+r})_{r\in\mathbf{R}}$, natural in r, for each $\sigma \in \Sigma$. A *homomorphism* from A to B is a natural transformation $h\colon A \to B$ such that

$$h_{\mathsf{g}(\sigma)+q} \cdot \sigma^A_q = \sigma^B_q(h_q \cdot -)$$

for all $q \in [0, \infty]$, where $h_r\colon A_r \to B_r$ denotes the component of h at $r \in \mathbf{R}$. We write $\mathbf{Alg}(\Sigma)$ for the category of Σ-algebras and their homomorphisms.

▶ **Example 4.2.** Let Σ_{Nb} be the signature with an operation symbol $\star_r$ of arity $\mathbf{2}_r$ and grade r for every $r \in [0, \infty]$. Then a Σ_{Nb}-algebra is a functor $A\colon \mathbf{R} \to \mathbf{pMet}$ together with a nonexpansive assignment of a point $(a \star^q_r b) \in A_{r+q}$ to each pair of r-neighbors in A_q.

Fix a signature Σ. We now describe a syntax which we use to specify full subcategories of $\mathbf{Alg}(\Sigma)$. We follow Kura [16] and employ a *coercion construct* $\mathsf{c}_{q\leq r}(-)$ to 'upgrade' terms along a morphism $q \leq r$ in $\mathbf{R}$. For each $r \in [0, \infty]$, the set $\mathscr{T}_{\Sigma,r}(X)$ of Σ-*terms of uniform depth* r (with variables in X) is inductively generated as follows:

1. $x \in \mathscr{T}_{\Sigma,0}(X)$ for each *variable* $x \in X$;
2. if $t_1, \ldots, t_n \in \mathscr{T}_{\Sigma,q}(X)$ and $\sigma \in \Sigma$ with $\mathsf{ar}(\sigma) = [n]$, then $\sigma(t_1, \ldots, t_n) \in \mathscr{T}_{\Sigma,\mathsf{g}(\sigma)+q}(X)$;
3. if $t \in \mathscr{T}_{\Sigma,q}X$ and $q \leq r$ in $[0, \infty]$, then $\mathsf{c}_{q\leq r}(t) \in \mathscr{T}_{\Sigma,r}(X)$.

▶ **Example 4.3.** For the signature Σ of Example 4.2, we have that $x \star_r y$ is a term of uniform depth r for *every* pair of variables $x, y \in X$ and $(x \star_r y) \star_s (x \star_r y)$ is a Σ-term of uniform depth $s + r$. The expression $(x \star_1 y) \star_0 (x \star_2 y)$ is not a uniform-depth term.

Given a map $\mathsf{ar}(\sigma) \to \mathscr{T}_{\Sigma,r}(X)$, we also write $\sigma(t) = \sigma(t_i)_{i\in\mathsf{ar}(\sigma)}$ if $t(i) = t_i$ for all $i \in \mathsf{ar}(\sigma)$. In this notation, we define the interpretation of terms in graded algebras as follows.

▶ **Definition 4.4.** A *valuation* of variables in a Σ-algebra A is a nonexpansive map $e\colon X \to A_q$. Every valuation extends to a family of partial maps $(e^\#_r\colon \mathscr{T}_{\Sigma,r}(X) \to A_{q+r})_{r\in\mathbf{R}}$ defined recursively on the structure of terms as follows:

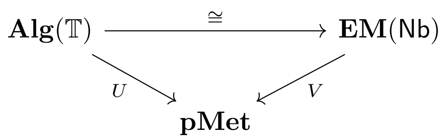
1. $e^{\#}_0(x) = e(x)$ for every variable $x \in X$;
2. $e^{\#}_{\mathsf{g}(\sigma)+r}(\sigma(t))$ is defined for $\sigma \in \Sigma$ and $t\colon \mathsf{ar}(\sigma) \to \mathscr{T}_{\Sigma,r}(X)$ iff $e^{\#}_r(t(i))$ is defined for all $i \in \mathsf{ar}(\sigma)$ and $e^{\#}_r \cdot t\colon \mathsf{ar}(\sigma) \to A_{q+r}$ is nonexpansive. In this case, $e^{\#}_{\mathsf{g}(\sigma)+r}(\sigma(t)) = \sigma^A_{q+r}(e^{\#}_r(t))$.
3. $e^{\#}_r(\mathsf{c}_{u\leq v}(t))$ is defined iff $e^{\#}_u(t)$ is defined, and $e^{\#}_r(\mathsf{c}_{u\leq v}(t)) = A(q+u \leq q+v)(e^{\#}_u(t))$.

▶ **Example 4.5.** Let us consider the signature of Example 4.2 once again. Given a valuation $e\colon X \to A_q$, the term $x \star_r y$ is evaluated by the partial map $e^{\#}_r\colon \mathscr{T}_{\Sigma,r}(X) \to A_{q+r}$. Unwinding definitions, we have that $e^{\#}_r(x_1 \star_r x_2)$ is defined precisely if $e^{\#}_0(x_i) = e(x_i)$ is defined for $i \in \{0,1\}$ (this is automatic since x_1, x_2 are variables) and $d_{A_q}(e(x_1), e(x_2)) \leq d_{\mathsf{ar}(\sigma)}(1,2)$.

▶ **Definition 4.6.** A *graded quantitative equality* is an expression of the form $\Gamma \vdash_r s =_u t$ where $\Gamma \in \mathbf{pMet}$ (the *context*) and $s, t \in \mathscr{T}_{\Sigma,r}(\Gamma)$. A Σ-algebra A *satisfies* $\Gamma \vdash_r s =_u t$ if, for every valuation $e\colon \Gamma \to A_q$, we have $e^{\#}_r(s)$ and $e^{\#}_r(t)$ are defined, and $d_{A_{q+r}}(e^{\#}_r(s), e^{\#}_r(t)) \leq u$. A *graded (qualitative) equality* has the form $\Gamma \vdash_r s = t$ and is satisfied by A if for every valuation $e\colon \Gamma \to A_q$ the maps $e^{\#}_r(s)$ and $e^{\#}_r(t)$ are defined and $e^{\#}_r(s) = e^{\#}_r(t)$. A *graded quantitative theory* is a set $\mathbb{T}$ of graded quantitative and qualitative equalities (the *axioms*). A $\mathbb{T}$-*model* is a Σ-algebra which satisfies every axiom of $\mathbb{T}$. We write $\mathbf{Alg}(\mathbb{T})$ for the full subcategory of $\mathbf{Alg}(\Sigma)$ spanned by all $\mathbb{T}$-models.

Graded quantitative theories are meant to be a formalism for generating $\mathbf{R}$-graded monads on the category $\mathbf{pMet}$. In order to make this precise, first note the assignment of a $\mathbb{T}$-algebra $A\colon \mathbf{R} \to \mathbf{pMet}$ to A_0 is the object-part of a *forgetful functor* $U\colon \mathbf{Alg}(\mathbb{T}) \to \mathbf{pMet}$ with the action on a homomorphism $h\colon A \to B$ defined by $U(h) = h_0\colon A_0 \to B_0$.

We leave it as an open problem to determine whether every graded quantitative theory $\mathbb{T}$ induces a graded monad $\mathbb{M}_{\mathbb{T}}$ such that $\mathbf{Alg}(\mathbb{T}) \cong \mathbf{EM}(\mathbb{M}_{\mathbb{T}})$ (as concrete categories), i.e., the triangle below commutes:



We conjecture that there is a positive resolution to this problem (cf. [9, 16]). On the other hand, we expect that a full characterization of the graded monads captured by our graded quantitative will be subtle because a characterization of the monads captured by ungraded basic quantitative theories [17] remains open [1, 3].

5 The Graded Quantitative Theory of Neighbors

We introduce the graded theory of neighbors inspired by the neighborhood monad. For ease of notation, we employ (finite) sets of expressions of the form $s =_r t$ as the contexts of axioms. These fully specifies a pseudometric space because $\mathbf{pMet}$ is a full epi-reflective subcategory of the category of relational structures and relation-preserving maps over the signature $\{=_r \mid r \in \mathbb{R}_\infty\}$ [9]. We blur the distinction between (finite) sets of relations and their reflections as (finite) spaces in $\mathbf{pMet}$.

▶ **Definition 5.1.** The *graded theory of neighborhoods* is the $\mathbf{R}$-graded quantitative theory $\mathbb{T}_{\mathsf{Nb}}$ whose signature consists of an operation $\star_r$ of arity $\mathbf{2}_r$ and grade r for each $r \in [0, \infty]$. These operations are subject to axiom

$\vdash_0 a = a \star_0 a$ *(idempotency)* and $\{a =_r b\} \vdash_s \mathsf{c}_{r\leq s}(a \star_r b) = a \star_s b$ *(merge)* (eliding the discrete context) and the *quantitative band laws* expressed as follows:

$$\{a =_r b, c =_r d, a =_s c\} \vdash_{r+s} (a \star_r b) \star_s (c \star_r d) = a \star_{r+s} d \quad \textit{(band decomposition)}$$
$$\{a =_r b, c =_r d, a =_s c\} \vdash_r a \star_r b =_s c \star_r d \quad \textit{(band distance)}$$

Axioms (merge), (band distance), and (band decomposition) are axiom schema.

The terminology 'quantitative band laws' stems from the rectangular bands of Clifford [7].

Let X be a pseudometric space. The assignment $r \mapsto T_r X$ is the object-part of a functor $\mathscr{F}(X)\colon \mathbf{R} \to \mathbf{pMet}$ with the action on a morphism $q \le r$ defined by the inclusion

$$\mathscr{F}(q \le r) := T_q X \hookrightarrow T_r X.$$

Indeed, preservation of identity morphisms and composition is clear, and $\mathscr{F}(q \le r)$ is defined because, for all $x, y \in X$, we have $d(a,b) \le q$ implies $d(a,b) \le r$ for all $r \ge q$. Note that $\mathscr{F}(X)$ carries the structure of a $\mathbb{T}_{\mathsf{Nb}}$-algebra (see Example 4.2) with the operations

$$\star_r^q\colon [\mathbf{2}_r, T_q X] \to T_{r+q} X, \qquad \star_r^q(f) := \langle \pi_1 \cdot f, \pi_2 \cdot f \rangle.$$

That is, for each pair of r-neighbors in $T_q X$ we have $(a,b) \star_r^q (c,d) := (a,d)$. These operations are nonexpansive being the composition of nonexpansive maps.

▶ **Proposition 5.2.** *$\mathscr{F}(X)$ is a $\mathbb{T}_{\mathsf{Nb}}$-algebra. In fact, $\mathscr{F}(X)$ is the free $\mathbb{T}_{\mathsf{Nb}}$-algebra with respect to the universal morphism $\eta\colon X \to (\mathscr{F}(X))_0$ defined by the assignment $a \mapsto (a,a)$.*

It follows from Proposition 5.2 that the assignment $X \mapsto \mathscr{F}(X)$ is the object-part of a functor $\mathscr{F}\colon \mathbf{pMet} \to \mathbf{Alg}(\mathbb{T}_{\mathsf{Nb}})$, and $\mathscr{F}$ is a left adjoint of the forgetful functor U.

Just as every monad arises from an adjunction, Fujii et al. [11, Section 3] explain that every $\mathbf{R}$-graded monad on a category $\mathscr{A}$ is induced by an adjunction $L \dashv R\colon \mathscr{B} \to \mathscr{A}$ and a strict action $\alpha\colon \mathbf{R} \times \mathscr{B} \to \mathscr{B}$ with underlying functors $M_r X := R(\alpha(r, LX))$. We apply this to the adjunction $\mathscr{F} \dashv U$ and the action given by equipping the functor $\alpha(r, A)(q) = A_{r+q}$ with the operations $(\star_q^u)(f\colon \mathbf{2}_q \to A_{r+u}) = \star_q^{r+u}(f)$. This yields an $\mathbf{R}$-graded monad $\mathbb{M}_{\mathbb{T}_{\mathsf{Nb}}}$ with $M_r X$ given by the r^{th}-component of the free $\mathbb{T}_{\mathsf{Nb}}$-algebra.

6 Concluding Remarks

We have introduced a concept of graded quantitative theory for generating $\mathbf{R}$-graded monads on the category **pMet** of pseudometric spaces and nonexpansive maps. The core concepts of our framework have been illustrated through the development of our graded quantitative theory of neighbors $\mathbb{T}_{\mathsf{Nb}}$. Our main result establishes the existence of free algebras in $\mathbf{Alg}(\mathbb{T}_{\mathsf{Nb}})$, which enabled us to associate an $\mathbf{R}$-graded monad $\mathbb{M}_{\mathbb{T}_{\mathsf{Nb}}}$ with $\mathbb{T}_{\mathsf{Nb}}$. While our primary motivation was to develop a new perspective on the neighborhood monad, we intend to further develop our general framework in future work. In particular, graded quantitative theories could be employed to characterize general properties (e.g. enrichment or accessibility rank) of $\mathbf{R}$-graded monads on **pMet** purely in terms of syntax. To this end, an important next step lies in the development of a graded quantitative equational logic for reasoning over graded quantitative theories.

In another direction, it remains open whether the graded quantitative theory of neighbors yields a presentation of the neighborhood monad. Once this has been established, we aim to systematically generate variants of the neighborhood monad with effects with applications to numerical analysis in mind. For instance, the extension of the graded theory of neighbors with the operations and equations of convex algebras may lead to methodologies for the forward error analysis of probabilistic programs.

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A Details for Section 5

Proof of Proposition 5.2. We first verify that $\mathscr{F}(X)$ is a $\mathbb{T}_{\mathsf{Nb}}$-algebra, i.e. it satisfies the axioms of $\mathbb{T}_{\mathsf{Nb}}$.

(idempotency): Unwinding the definition of satisfaction, we aim to show that for each $q \in \mathbf{R}$ and each point $(x, y) \in (\mathscr{F}(X))_q = T_q X$ we have $(a, b) \star_q (a, b) = (a, b)$. This is automatic by the definition of $\star_q$.

(merge): We want to show that for and r-neighbors $(a, b), (c, d) \in (\mathscr{F}(X))_q$ for $q \in \mathbf{R}$ we have $\mathsf{c}_{r\leq s}((a, b) \star^q_r (c, d)) = (a, b) \star^q_s (c, d)$. To this end, we first note that $(a, b) \star^q_r (c, d)$ is defined since $(a, b), (c, d)$ are r-neighbors. Thus, by unfolding Definition 4.4, we have that

$$\mathsf{c}_{r\leq s}((a, b) \star^q_r (c, d)) = \mathscr{F}(q + r \leq q + s)((a, b) \star^q_r (c, d)) = (a, b) \star^q_s (c, d),$$

as desired, because $\mathscr{F}(X)(q + r \leq q + s)$ is the inclusion of $(\mathscr{F}(X))_{q+r}$ into $(\mathscr{F}(X))_{q+s}$.

(band distance): Given a valuation $f\colon \{a =_r b, c =_r d, a =_s c\} \to (\mathscr{F}(X))_q$, we have that

$$(\mathscr{F}(X))_q \models f(a) =_r f(b) \quad \text{and} \quad (\mathscr{F}(X))_q \models f(c) =_r f(d)$$

because f is nonexpansive. In particular, $e^\#_{q+r}(f(a) \star_r f(b))$ and $e^\#_{q+r}(f(c) \star_r f(d))$ are defined. Observe that

$$e^\#_{q+r}(f(a) \star_r f(b)) = f(a) \star^{q+r}_r f(b) = (\pi_1(f(a)), \pi_2(f(b)) \in (\mathscr{F}(X))_{q+r}$$

and, similarly, $f(c) \star^{q+r}_r f(d) = (\pi_1 f(c), \pi_2(f(d)))$. With this on hand, note that we have

$$\begin{aligned} d(e^\#_{q+r}(f(a) \star_r f(b)), e^\#_{q+r}(f(a) \star_r f(b))) &= d((\pi_1(f(a)), \pi_2(f(b)), (\pi_1(f(c)), \pi_2(f(d))) \\ &= d_X(\pi_1(f(a)), \pi_1(f(c))) \\ &\leq s \end{aligned}$$

where we have used the definition of the valuation of terms, the definition of the pseudometric on $T_{q+r}X$, and that the composite map $\pi_1 \cdot f$ is nonexpansive, respectively. It follows that $\mathscr{F}(X)$ satisfies (band distance).

(band decomposition): As before, let $f\colon \{a =_r b, c =_r d, a =_s c\} \to (\mathscr{F}(X))_q$ be a valuation. We see by the same argument as for (band distance) that the left hand side subterms $e^\#_{q+r}(f(a) \star_r f(b))$ and $e^\#_{q+r}(f(c) \star_r f(d))$ are defined. Then, in order to see the left hand side is defined, note that we have

$$e^\#_{q+r+s}((f(a) \star_r f(b)) \star_s (f(c) \star_r f(d))) = e^\#_{q+r}(f(a) \star_r f(b)) \star^{q+r}_s e^\#_{q+r}(f(c) \star_r f(d))$$

Definedness of this term follows by application of (band distance) ensuring the subterms are s-neighbors. Similarly the right hand side is defined by the triangle inequality:

$$\begin{aligned} d(e^\#_{q+r+s}(f(a) \star_{r+s} f(d))) &= d_X(\pi_1 f(a), \pi_2 f(d)) \\ &\leq d_X(\pi_1 f(a), \pi_1 f(c)) + d_X(\pi_1 f(c), \pi_1 f(d)) + d_X(\pi_1 f(d), \pi_2 f(d)) \\ &= d(a, c) + d(c, d) + d_X(\pi_1 f(d), \pi_2 f(d)) \\ &\leq s + r + q \end{aligned}$$

Given that the terms are defined, the desired equality follows from the definition of $\star$:

$$\begin{aligned} e^{\#}_{q+r+s}((f(a)\star_r f(b))\star_s(f(c)\star_r f(d))) &= (\pi_1(f(a)),\pi_2 f(b))\star_s^{q+r}(\pi_1(f(c)),\pi_2 f(d)) \\ &= (\pi_1(f(a)),\pi_2 f(d)) \\ &= e^{\#}_{q+r+s}(f(a)\star_{r+s} f(d)) \end{aligned}$$

We conclude that $\mathscr{F}(X)$ satisfies (band decomposition).

We may now conclude that $\mathscr{F}(X)$ sits in $\mathbf{Alg}(\mathbb{T}_{\mathsf{Nb}})$, as desired. It remains to show that $\mathscr{F}(X)$ has the structure of a free $\mathbb{T}_{\mathsf{Nb}}$-algebra with respect to the universal morphism η and the forgetful functor U that takes 0-parts. This means that for any valuation $h\colon X \to A_0$ of the variables from X in a $\mathbb{T}_{\mathsf{Nb}}$-algebra A, there exists a unique homomorphism $h^{\#}\colon \mathscr{F}(X) \to A$ such that $h = h_0^{\#}\cdot\eta$.

To this end, consider the map $\bar{h}_q\colon T_q X = (\mathscr{F}(X))_q \to A_q$ defined for each $q \in \mathbf{R}$ by

$$\bar{h}_q((a,b)) := h(a)\star_q^0 h(b).$$

We proceed to show that $\bar{h}$ is a homomorphism $\mathscr{F}(X)$ to A. To this end, first observe that for the valuation h, the extension $h_q^{\#}\colon \mathscr{T}_{\Sigma,q}(X) \to A_q$ is defined on $a\star_q b$ if $(a,b) \in (\mathscr{F}(X))_q = T_r X$: indeed, $A_q \models h(a) =_r h(b)$ because h is nonexpansive. It follows that $h_r^{\#}(\mathsf{c}_{q\le r}(a\star_q b)) = A(q\le r)(h(a)\star_q^0 h(b))$ is defined for all $q \le r$ in $\mathbf{R}$. In particular, since A is a $\mathbb{T}_{\mathsf{Nb}}$-algebra, it follows from (merge) that

$$A(q\le r)(h(a)\star_q^0 h(b)) = h(a)\star_r^0 h(b) \tag{1}$$

for all $(a,b) \in (\mathscr{F}(X))_q$. With this on hand, we may now show that the maps $\bar{h}_q$ are the components of a natural transformation $h\colon \mathscr{F}(X) \to A$, i.e. the following square commutes for all $q \le r$ in $\mathbf{R}$:

$$\begin{array}{ccc} (\mathscr{F}(X))_q & \xrightarrow{\mathscr{F}(X)(q\le r)} & (\mathscr{F}(X))_r \\ \Big\downarrow{\scriptstyle \bar{h}_q} & & \Big\downarrow{\scriptstyle \bar{h}_r} \\ A_q & \xrightarrow{A(q\le r)} & A_q \end{array}$$

Indeed, we compute as follows

$$\begin{aligned} \bar{h}_q\cdot\mathscr{F}(X)(q\le r)((a,b)) &= \bar{h}_r((a,b)) && \text{(by defn. of } \mathscr{F}(X)(q\le r)) \\ &= h(a)\star_r^0 h(b) && \text{(by defn. of } \bar{h}_r) \\ &= A(q\le r)(h(a)\star_q^0 h(b)) && \text{(by (1))} \\ &= A(q\le r)\cdot\bar{h}_q((a,b)) && \text{(by defn. of } \bar{h}_q) \end{aligned}$$

It remains to show that

$$\bar{h}_{r+q}(a\star_r^q b) = \bar{h}_q(a)\star_r^q \bar{h}_q(b) \tag{2}$$

whenever $(\mathscr{F}(X))_q \models a =_r b$. In other words, we have $d_X(\pi_1(a),\pi_1(b)) \le r$. First, unfold the left-hand side of (2) as follows:

$$\bar{h}_{r+q}(a\star_r^q b) = \bar{h}_{r+q}((\pi_1(a),\pi_2(b)) = h(\pi_1(a))\star_{r+q}^0 h(\pi_2(b)) \tag{3}$$

Now, we compute as follows from the right- to left-hand side of (2):

$$\begin{aligned}\bar{h}_q(a) \star_r^q \bar{h}_q(b) &= (h(\pi_1(a)) \star_q^0 h(\pi_2(a))) \star_r^q (h(\pi_1(b)) \star_q^0 h(\pi_2(b))) \\ &= h(\pi_1(a)) \star_{r+q}^0 h(\pi_2(b)) && \text{(by (band decomposition))}\end{aligned}$$

Thus, $\bar{h}$ is a homomorphism of Σ_{Nb}-algebras,

Finally, we show that $\bar{h}$ satisfies the universal mapping property described above. To this end, we compute as follows:

$$\bar{h}_0(\eta(x)) = \bar{h}_0((x,x)) = h(x) \star_0^0 h(x) = h(x)$$

where we use (idempotency) in the last step. Further, if $\bar{g}\colon \mathscr{F}(X) \to A$ is a homomorphism with $g_0 \cdot \eta = h$, we have

$$\begin{aligned}\bar{g}_r((a,b)) &= \bar{g}_r((a,a) \star_r^0 (b,b)) && \text{(by defn. of } \star_0^r\text{)} \\ &= \bar{g}_0((a,a)) \star_r^0 \bar{g}_0((b,b)) && \text{(since } \bar{g} \text{ is a homom.)} \\ &= h(a) \star_r^0 h(b) && \text{(since } \bar{g}_0 \cdot \eta = h\text{)} \\ &= \bar{h}_r((a,b)) && \text{(by defn. of } \bar{h}_r\text{)}\end{aligned}$$

We conclude that $\bar{h}$ is the unique homomorphism with $\bar{h}_0\eta = h$. It now follows that $\mathscr{F}(X)$ is a free $\mathbb{T}_{\mathsf{Nb}}$-algebra on X, as desired. ◀